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LETTER TO THE EDITOR

Spectral statistics for quantized skew translations on the torus

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Abstract. We study the spectral statistics for quantized skew translations on the torus, which are ergodic but not mixing for irrational parameters. It is shown explicitly that in this case the level-spacing distribution and other common spectral statistics, like the number variance, do not exist in the semiclassical limit.

Introduction

One of the central questions in quantum chaos is how the asymptotic distribution of the energy levels of a quantum system depends on the behaviour of the corresponding classical dynamical system. For integrable systems the spectral statistics have been conjectured [1] to be Poissonian, whereas chaotic systems have been conjectured [2] to be described by random matrix theory (like the Gaussian orthogonal ensemble for systems with time-reversal symmetry). Both conjectures are supported by many numerical studies. However, in both cases exceptions are known: so-called arithmetic systems (see, e.g., [3–7]) show Poissonian spectral statistics despite being strongly chaotic. Another example showing non-generic spectral statistics are quantized cat maps [8, 9]. As an example of a class of integrable systems the eigenvalue statistics for flat tori are studied in [10]. It is proven that the pair correlation function is Poissonian for a set of full Lebesgue measures in the parameter space of tori, but that it does not exist for a set of second Baire categories (a topologically large set). Explicit examples of tori with a Poissonian pair correlation function are given in [11]. A further class of integrable systems showing exceptional behaviour are harmonic oscillators, for which the nearest-neighbour level spacing and other spectral statistics do not possess a limit distribution, see e.g. [1, 12–16] and references therein.

An important class of model systems for studies in quantum chaos arise from the quantization of area-preserving maps, see e.g. [8, 17] and references therein. In this paper we study the spectral statistics, i.e. the distribution of eigenphases, for the class of quantized skew translations on the torus (also called parabolic maps) [18–20].

Spectral statistics

A particular example of a skew translation on the torus $\mathbb{T}^2$ (see e.g. [21]) is defined by

$$\begin{pmatrix} p \\ q \end{pmatrix} \stackrel{A_\alpha}{\rightarrow} \begin{pmatrix} p+\alpha \\ q+2p \end{pmatrix} \qquad \text{mod } 1 \tag{1}$$

where $\alpha \in \mathbb{R}^+$ determines the dynamical behaviour: for rational α the mapping is not ergodic, whereas for irrational α the map is ergodic and, in particular, uniquely ergodic [22], i.e. there is only one invariant ergodic measure, a situation rarely encountered for a dynamical system. This implies that A_α does not possess any periodic points for α irrational. Moreover, A_α is not weakly mixing and thus also not mixing, see e.g. [21].

A quantization of an area-preserving map on the torus is given by a sequence of unitary time evolution operators U_N defined on an N-dimensional Hilbert space, where $N \to \infty$ corresponds to the semiclassical limit. For quantization of these skew translations we use the one proposed in [20], which is based on considering appropriate rational approximations a_N/N to α. That is, for a given irrational α and $N \in \mathbb{N}$ there is a unique $a_N \in \mathbb{N}$ defined by the condition

$$\left|\alpha - \frac{a_N}{N}\right| < \frac{1}{2N}. \tag{2}$$

Then the propagator U_N can be expressed in the position representation by the $N \times N$ unitary matrix

$$(U_N)_{kj} = \frac{1}{N}\sum_{l=0}^{N-1} \exp\left(\frac{2\pi \mathrm{i}}{N}(lk - (l-a_N)^2 - (l-a_N)j)\right) \tag{3}$$

with $j, k \in \{0, 1, \ldots, N-1\}$. One investigates the eigenvalues $\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\phi_j}$ of U_N, where $\phi_j \in [0, N[$ and $j \in \{0, \ldots, N-1\}$. The spectral density $\varrho(\phi)$ is given by

$$\varrho(\phi) := \sum_{j=0}^{N-1}\sum_{k\in\mathbb{Z}} \delta\left(\frac{2\pi}{N}(\phi - \phi_j) - 2\pi k\right) \tag{4}$$

and using the Poisson summation formula $\varrho(\phi)$ can be expressed in terms of U_N by

$$\varrho(\phi) = \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}l\phi}\,\mathrm{Tr}\,U_N^l. \tag{5}$$

For the skew translations the eigenphases of the matrix U_N can be determined explicitly [20]:

$$\phi_{\eta,l} = lD - \eta^2 + \eta a_N - a_N^2\frac{(M-1)(2M-1)}{6} \mod N \tag{6}$$

with $\eta \in \{1, \ldots, D\}$, $l \in \{0, \ldots, M-1\}$ and $M = N/D$, where $D = \gcd(a_N, N)$ is the greatest common divisor of a_N and N.

An important statistics is the level-spacing distribution, which is the probability density for the distribution of the distances $\phi_{j+1} - \phi_j$ between (unfolded) eigenphases $\phi_j \in [0, N[$. More precisely, one considers (with $\phi_N := \phi_0$)

$$\lim_{N\to\infty} \frac{\#\{j < N \mid a \leqslant \phi_{j+1} - \phi_j \leqslant b\}}{N} = \int_a^b P(s)\,\mathrm{d}s \tag{7}$$

if a limit distribution $P(s)$ exists. From equation (6) follows $\phi_{\eta,l} + D = \phi_{\eta,l+1}$ and consequently the spectrum is periodic with period D. Moreover, the last term in equation (6) is independent of η and l such that for the level-spacing distribution it is sufficient to study the reduced spectrum

$$\varphi_\eta := -\eta^2 + \eta a_N \mod D = -\eta^2 \mod D \tag{8}$$

with $\eta \in \{1, \ldots, D\}$. For a fixed $\alpha \in \mathbb{R}^+$ and a given N, equation (2) fixes a rational approximant $a_N \in \mathbb{N}$ and also $D = \gcd(a_N, N)$. Let us consider three special cases. First assume that $D = 1$. Then the reduced spectrum equation (8) consists of just one number, i.e. the original spectrum (6) is completely rigid, leading to a level-spacing distribution

$$P_{D=1}(s) = \delta(s-1). \tag{9}$$

Assuming $D = 2$ we get for the reduced spectrum

$$\varphi_1 = -1 \bmod 2 \equiv 1 \qquad \text{and} \qquad \varphi_2 = -4 \bmod 2 \equiv 0. \tag{10}$$

Thus the spectrum equation (6) is composed of two subsequences $\phi_{1,l}$, $\phi_{2,l}$ with an equidistant spacing of $D = 2$. Since these two subsequences are shifted with respect to each other by $3 \bmod 2 \equiv 1$, we obtain for the level-spacing distribution $P_{D=2}(s) = \delta(s-1)$ as in the case where $D = 1$. Finally, we consider the special case $D = 3$. The reduced spectrum is given by

$$\varphi_1 = -1 \bmod 3 \equiv 2 \qquad \varphi_2 = -4 \bmod 3 \equiv 2 \qquad \text{and} \qquad \varphi_3 = -9 \bmod 3 \equiv 0. \tag{11}$$

Thus the spectrum (6) consists of three subsequences. Two of them, $\phi_{1,l}$ and $\phi_{2,l}$, lead to the same eigenphases, i.e. the spacing between them is zero. The spacing between these two subsequences and the third subsequence is 2 and 1, respectively. Thus we get for the level-spacing distribution

$$P_{D=3}(s) = \tfrac{1}{3}[\delta(s) + \delta(s-1) + \delta(s-2)]. \tag{12}$$

Using the cases of $D = 1$ and $D = 3$ we show that there is no limit distribution of the level-spacing distribution for the quantized skew translations in the limit $N \to \infty$ by an explicit construction of two different limit points of the sequence of level-spacing distributions.

A general result from the approximation theory of irrational numbers, see e.g. [23], asserts that for any irrational α there exists an infinite sequence of pairs (a_N, N) with $a_N, N \in \mathbb{N}$ and $\gcd(a_N, N) = 1$ such that

$$\left|\alpha - \frac{a_N}{N}\right| < \frac{1}{N^2}. \tag{13}$$

All these pairs are approximations fulfilling equation (2). If (a_N, N) is such an approximation then $(a_{N'}, N') = (D'a_N, D'N)$ for $N \geqslant 2D'$ is also a good approximation. This follows from

$$\left|\alpha - \frac{a_{N'}}{N'}\right| = \left|\alpha - \frac{a_N}{N}\right| < \frac{1}{N^2} \leqslant \frac{1}{2D'N} = \frac{1}{2N'}. \tag{14}$$

This implies that for each $D \in \mathbb{N}$ there is an infinite sequence of pairs (a_N, N) with $D = \gcd(a_N, N)$ fulfilling equation (2). sequence of D values in the limit $N \to \infty$. With the explicit calculation of the level-spacing distribution $P(s)$ for $D = 1$ and $D = 3$ we obtain two infinite sequences for which the level-spacing distributions are different. Consequently there is no limit of the level-spacing distribution as $N \to \infty$.

Another commonly used statistics is the number variance which measures long-range correlations in the spectrum. For quantized maps with unfolded eigenphases $\phi_j \in [0, N[$ the number variance is defined by

$$\Sigma^2(L; N) := \frac{1}{N}\int_0^N (\mathcal{N}(\phi + L) - \mathcal{N}(\phi) - L)^2 \, \mathrm{d}\phi \tag{15}$$

where $\mathcal{N}(\phi) := \int_0^\phi \varrho(\phi')\, \mathrm{d}\phi'$ is the integrated spectral density. Notice that for $L \leqslant N$ we have $\Sigma^2(L, N) = \Sigma^2(N - L, N)$.

Using equation (4) the number variance can be expressed in terms of the propagator U_N

$$\Sigma^2(L; N) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(\frac{n\pi L}{N}\right)|\operatorname{Tr} U_N^n|^2. \tag{16}$$

From the explicit expression (6) for the eigenphases one obtains

$$\operatorname{Tr} U_N^n = \sum_{\eta=1}^{D}\sum_{l=0}^{M-1}\exp(n\phi_{\eta,l})$$

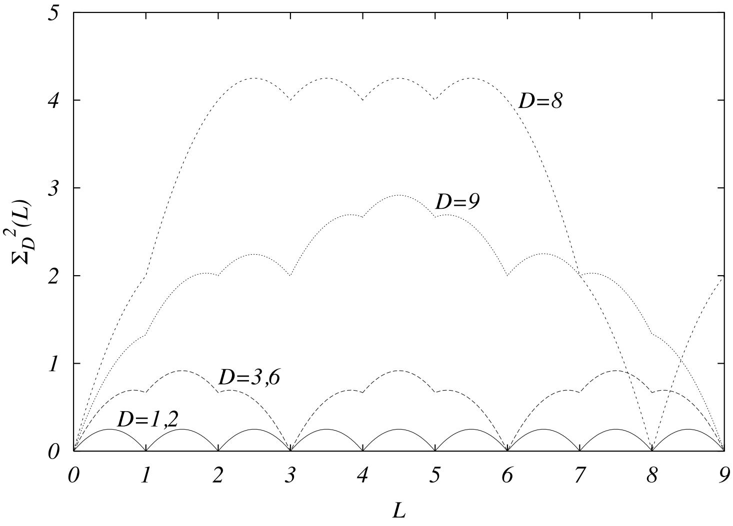

Figure 1. Number variance for the quantized skew translations on the torus for $D = 1, 2, 3, 6, 8$ and $D = 9$.

$$= \sum_{\eta=1}^{D} \sum_{l=0}^{M-1} \exp\left(\frac{2\pi \mathrm{i}}{N} n \left(lD - \eta^2 + \eta a_N - a_N^2 \frac{(M-1)(2M-1)}{6}\right)\right)$$

$$= M\delta_{(n \bmod M),0} \sum_{\eta=1}^{D} \exp\left(\frac{2\pi \mathrm{i}}{N} n \left(-\eta^2 + \eta a_N - a_N^2 \frac{(M-1)(2M-1)}{6}\right)\right). \quad (17)$$

This implies for the number variance of quantized skew maps with $D = \gcd(a_N, N)$

$$\Sigma_D^2(L) = \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin^2\left(\frac{k\pi L}{D}\right) \left| \sum_{\eta=1}^{D} \exp\left(-\frac{2\pi \mathrm{i}}{D} k\eta^2\right)\right|^2. \quad (18)$$

Notice, that $\Sigma_D^2(L)$ does not depend explicitly on a_N and N, but only on their greatest common divisor D. For $D = 1$ we get

$$\Sigma_{D=1}^2(L) = \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin^2(k\pi L) = (L - \lfloor L \rfloor) + (L - \lfloor L \rfloor)^2 \quad (19)$$

where $\lfloor x \rfloor$ denotes the integer part of x. The same result also holds for $D = 2$. In the case of $D = 3$ the computation of the Fourier series involved leads to

$$\Sigma_{D=3}^2(L; N) = -\frac{8}{9} + 5F\left(\frac{L}{3}\right) + 2F\left(\frac{L-2}{3}\right) + 2F\left(\frac{L+2}{3}\right) \quad (20)$$

where we defined $F(x) := x - \lfloor x \rfloor + (x - \lfloor x \rfloor)^2$. Thus the number variance is different for $D = 1$ and $D = 3$, and consequently there is also no limit of the number variance as $N \to \infty$.

In figure 1 we show four examples for the behaviour of the number variance in dependence on D. The cases $D = 2$ and $D = 6$ coincide with $D = 1$ and $D = 3$ respectively, which illustrates that D is not necessarily the smallest period of $\Sigma_D^2(L; D)$. A higher number of degeneracies in the reduced spectrum equation (8), as for example in the case of $D = 8$, leads to large values of the number variance.

Discussion

The non-existence of limit distributions for the spectral statistics of quantized skew translations provides another counterexample to the universality of energy-level statistics observed in many situations. In contrast to the case of flat tori one has for the class of quantized skew transformations explicit examples for which the spectral statistics do not exist. There are different ways to interpret this result. On the one hand, this example may be seen as an indication that in order to obtain the expected random matrix behaviour not just ergodicity but also the mixing property of the classical system is needed. On the other hand, one may consider this class of systems as being quite non-generic, in a similar manner as the quantized cat maps. Finally, we would like to remark that it may be possible that certain spectral statistics exist for $N \to \infty$ when one averages over a (possibly increasing) range of different N, as was shown for the quantized cat maps [9].

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